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ON AN OPEN PROBLEM OF S. OWA

by

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Let U denote the unit disc, $U=\{z\in \mathbf{C};|z|<1\}$, let $\mathbf{N}$ denote the set of positive integers, $\mathbf{N}=\{1,2,3,\ldots\}$ and let $H(U)$ denote the set of functions which are holomorphic in U.

For $n\in \mathbf{N}$ let

$$T_{n}=\left\{f\in H(U);\frac{f(z)}{z}\neq 0,(z\in \mathbf{C}-\{0\}),f(z)=z-\sum_{k=n+1}^{\infty}a_{k}z^{k},a_{k}\geq 0,(k\in \mathbf{N},k>n)\right\}.$$

For $n\in \mathbf{N}$ and $b\in \mathbf{C}-\{0\}$ we define the next subclasses of T_{n}

$$T_{n}^{*}(b)=\left\{f\in T_{n}:\mathrm{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0,(z\in U)\right\},$$

$$O_{n}^{*}(b)=\left\{f\in T_{n}:\sum_{k=n+1}^{\infty}(k-1+|b|)a_{k}\leq|b|\right\}$$

and

$$P_{n}^{*}(b)=\left\{f\in T_{n}:\sum_{k=n+1}^{\infty}\left[(k-1)\frac{\mathrm{Re}\, b}{|b|}+|b|\right]a_{k}\leq|b|\right\}.$$

The functions in $T_{n}^{*}(b)$ are the functions with negative coefficients starlike of the complex order b (see [1, 2]).

The class $T_{1}^{*}(1-\alpha)$, $\alpha\in[0,1)$ is the class of functions with negative coefficients starlike of order α introduced and studied by H. Silverman [4].

The class $O_{n}^{*}(b)$ was introduced by S. Owa in [3, p.163-164], where he conjectured that $T_{n}^{*}(b)=O_{n}^{*}(b)$. In this paper we give an answer to this conjecture.

THEOREM .*Let* $n\in \mathbf{N}$ *and let* $b\in \mathbf{C}-\{0\}$; *then*

1) $O_{n}^{*}(b)\subset T_{n}^{*}(b)$;

2) $T_{n}^{*}(b)\subset P_{n}^{*}(b)$;

3) *If* $b \in (0, \infty)$ *(*b *is a positive real number), then*

$$O_n^*(b) = T_n^*(b) = P_n^*(b);$$

4) *If* $-n/2 < \mathbf{Re}\, b \leq 0$, *then* $P_n^*(b) \not\subseteq T_n^*(b)$;

5) *If* $b \in (-\infty, -n) \cup (-n/2, 0)$, *then* $T_n^*(b) \not\subseteq O_n^*(b)$.

Proof. 1). Let $f \in O_n^*(b)$. We prove that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < |b|, \quad z \in U. \tag{1}$$

We suppose that f has the series expansion

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{2}$$

We have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| - |b| = \left| \frac{\sum_{k=n+1}^{\infty} (k-1) a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k z^{k-1}} \right| - |b| \leq \frac{\sum_{k=n+1}^{\infty} (k-1) a_k |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k |z|^{k-1}} - |b|. \tag{3}$$

We use the fact that $f(z) \neq 0$ when $z \in U - \{0\}$ and $\lim_{z \to 0} [f(z)/z] = 1$; these imply

$$1 - \sum_{k=n+1}^{\infty} a_k |z|^k > 0, \tag{4}$$

when $|z| = r \in [0, 1]$.

From (3) and (4) we deduce

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| - |b| < \frac{\sum_{k=n+1}^{\infty} (k-1) a_k}{1 - \sum_{k=n+1}^{\infty} a_k} - |b|$$

$$= \frac{\sum_{k=n+1}^{\infty} (k - 1 + |b|) a_k - |b|}{1 - \sum_{k=n+1}^{\infty} a_k}.$$

By using the definition of $O_n^*(b)$ we obtain (1) and this implies

$$\mathrm{Re} \left\{ \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > -1, \quad z \in U,$$

hence $f \in T_n^*(b)$.

2). Let f be in $T_n^*(b)$. Then

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0 \quad (z\in U)$$

and, by using (2), this is equivalent to

$$\operatorname{Re}\left\{\frac{1}{b}\frac{\sum_{k=n+1}^{\infty}(1-k)a_k z^{k-1}}{1-\sum_{k=n+1}^{\infty}a_k z^{k-1}}\right\}>-1 \quad (z\in U). \tag{5}$$

If $z=r\in[0,1)$ and for $r\to 1^-$, from (5) we obtain

$$\frac{\sum_{k=n+1}^{\infty}(1-k)a_k}{1-\sum_{k=n+1}^{\infty}a_k}\operatorname{Re}\frac{1}{b}>-1$$

wich is equivalent to

$$\sum_{k=n+1}^{\infty}\operatorname{Re}b\,(1-k)a_k>-|b|^2\left(1-\sum_{k=n+1}^{\infty}a_k\right)$$

or

$$\sum_{k=n+1}^{\infty}\left[(k-1)\operatorname{Re}b\,/|b|+|b|\right]a_k<|b|\,,$$

hence $f\in P_n^*(b)$.

3). If b is a real positive number, then the definition of O_n^* and P_n^* are equivalent, hence $O_n^*(b)=P_n^*(b)$. By using 1) and 2) from this theorem we obtain the conclusion of 3).

4). Let

$$f_n(z)=z-z^{n+1}\,; \tag{6}$$

then $f_n\in P_n^*(b)$ when $b\in\mathbf{C}-\{0\}$ and $\operatorname{Re}b<0$, because

$$\sum_{k=n+1}^{\infty}\left[|b|+((k-1)\operatorname{Re}b\,)/|b|\right]a_k$$

$$=\{|b|+[(n+1)-1]\operatorname{Re}b\,/|b|\}\cdot 1=|b|+n\operatorname{Re}b\,/|b|\le|b|\,.$$

Now let $\rho=\operatorname{Re}b<0$ and let s be a negative real number such that

$$n+2\rho(1-s)>0$$

for $n \in \mathbf{N}$ fixed. If we choose z_0 one of the rooth of the equation

$$z^n = \frac{b(1-s)}{n+b(1-s)},$$

then $z_0 \in U$ and for f_n given by (6) we have

$$1 + \frac{1}{b}\left(\frac{z_0 f_n'(z_0)}{f_n(z_0)} - 1\right) = s < 0,$$

hence $f_n \notin T_n^*(b)$.

5). Let $b \in (-\infty, -n)$; we verify that the functions

$$f_{n,\lambda}(z) = z - \lambda z^{n+1} \tag{7}$$

belong to $T_n^*(b)$ for $\lambda > b/(n+b)$ and that $f_{n,\lambda} \notin O_n^*(b)$.

Indeed we have

$$\sum_{k=n+1}^{\infty} (k-1+|b|)a_k = (n+|b|)\lambda > |b|,$$

because $\lambda > b/(n+b) > 1$.

We also have

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z f_{n,\lambda}'(z)}{f_{n,\lambda}(z)} - 1\right)\right\} = \operatorname{Re}\left\{1 + \frac{n\lambda z^n}{b(\lambda z^n - 1)}\right\} > 0\ , z \in U, \tag{8}$$

for $\lambda > b/(n+b)$ and $b < -n$, hence $f_{n,\lambda} \in T_n^*(b)$.

Let now $b \in (-n/2, 0)$, and let $f_{n,\lambda}$ be defined by (7), where

$$-b/(n-b) < \lambda < -b/(n+b).$$

Then $\lambda > -b/(n-b)$ implies $f_{n,\lambda} \notin O_n^*(b)$ and for $\lambda < -b/(n+b)$, $-n/2 < b < 0$ the inequality (8) also is verified, hence $f_{n,\lambda} \in T_n^*(b)$..

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